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SMALL VIBRATIONS OF PRESTRAINED CONSTRAINED ELASTIC MATERIALS: THE IDEALIZED FIBRE-REINFORCED MATERIAL

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Abstract—We derive the general equations satisfied by small vibrations of arbitrary form superimposed upon a finite, static deformation (not necessarily homogeneous) of an elastic body of arbitrary anisotropy suffering an unspecified number of constraints of fully general form. Specialization is then made to a fibre-reinforced material, modelled here as an incompressible material that is inextensible in the fibre direction. The slowness surface is a one-sheeted, centro-symmetric surface except that two slownesses are possible for waves travelling along, or normal to, the fibre-direction. In many of these exceptional directions the slowness surface exhibits singular behaviour which is fully discussed. Numerical illustrations are presented.

1. INTRODUCTION

There is currently great theoretical interest in the mechanical properties of fibre-reinforced materials, presumably because of their ever increasing application in the engineering field. Here we shall study wave propagation in such materials, taking as our model that of an incompressible isotropic matrix reinforced by inextensible fibres. The continuum model is therefore that of an incompressible hyperelastic solid that is inextensible in a specified direction (the fibre direction) and in addition is transversely isotropic about that direction. Such a material is often termed an idealized fibre-reinforced material. Spencer (1972) has written an excellent general account of such materials. Weitsman (1972) and Chen and Gurtin (1974) discussed wave propagation in materials that are inextensible but not incompressible whilst Scott and Hayes (1976) appear to have been the first to consider wave propagation in idealized fibre-reinforced materials as presently defined.

Leaving aside the specific constraints of incompressibility and inextensibility for the moment, we recall that the present author [see Scott (1975)] has given an account of acceleration wave propagation in constrained elastic materials in which the constraints assume a fully general form. In the same spirit Chadwick *et al.* (1985) have given an analysis of small-amplitude waves in a homogeneously prestrained elastic body suffering an unspecified number of constraints of fully general form. In many respects we follow their analysis here.

The first part of Section 2 is devoted to the derivation of the equations [see (11) below] of small vibrations of arbitrary form superimposed upon a finite, static deformation (not necessarily homogeneous) of an elastic body of arbitrary anisotropy suffering an unspecified number of constraints of fully general form. This extends the work of Toupin and Bernstein (1961), who did not include the effects of constraints, and that of Scott and Hayes (1976), who permitted non-homogeneous prestrain but considered only the idealized fibre-reinforced material. Chadwick *et al.* (1985) considered fully general constraints but took the prestrain to be homogeneous. This level of generality is not, however, maintained in the following where specialization is made to small-amplitude sinusoidal wave propagation in a homogeneously prestrained and prestressed doubly-constrained elastic solid.

In Section 3 further specialization is made and we discuss the slowness surface of an idealized fibre-reinforced material. For non-exceptional directions **n** of wave propagation this is a one-sheeted, centro-symmetric surface with the property that any plane cross-section containing the fibre direction **a** is elliptical. Waves propagating (i) parallel to, or (ii) normal to, the fibre direction are found to be *exceptional* in that *two* waves may propagate in any such direction. In all cases it is shown that the two (generally distinct) wave speeds in an exceptional direction bound the possible wave speeds obtainable as limiting cases of wave propagation in non-exceptional directions.

In the final section numerical illustrations of the preceding theory are discussed.

2. BASIC EQUATIONS AND PROPAGATION CONDITIONS

We consider an elastic body $\mathscr{B}$ which is subject to one or more internal constraints and possesses a natural undistorted state B_0 and an equilibrium configuration $\bar{B}$ which is taken as the reference state. The configuration at the current time t is denoted by B_t and throughout the following is assumed to differ only infinitesimally from $\bar{B}$ at all times. The position of a generic particle in B_0, $\bar{B}$ and B_t is denoted by $\mathbf{X}$, $\bar{\mathbf{x}}$ and $\mathbf{x}$, respectively. The assumptions that $\bar{B}$ is an equilibrium state and that B_t differs only infinitesimally from $\bar{B}$ imply that

$$\bar{\mathbf{x}} = \bar{\mathbf{x}}(\mathbf{X}), \quad \mathbf{x} = \bar{\mathbf{x}} + \varepsilon\mathbf{u}(\bar{\mathbf{x}}, t), \tag{1}$$

where $0 < \varepsilon \ll 1$ and $\varepsilon\mathbf{u}$ is the particle displacement from $\bar{B}$ to B_t. The gradients of the deformations $B_0 \to \bar{B}$ and $B_0 \to B_t$ are written respectively as

$$\bar{F}_{iA} = \frac{\partial \bar{x}_i}{\partial X_A}, \quad F_{iA} = \frac{\partial x_i}{\partial X_A} = \bar{F}_{iA} + \varepsilon u_{i,j}\bar{F}_{jA}, \tag{2}$$

the last equation following from $(1)_2$ and the chain rule of partial differentiation. Repeated suffices are summed over. The notation $(\ )_{,j}$ is used exclusively to mean the partial derivative $\partial(\ )/\partial\bar{x}_j$ and never $\partial(\ )/\partial x_j$. We denote the Jacobian determinants of the deformations $\bar{\mathbf{F}}$ and $\mathbf{F}$ in (2) by $\bar{J}$ and J, respectively, and the densities measured in $\bar{B}$ and B_t by $\bar{\rho}$ and ρ, respectively. On utilizing mass conservation in the form $\bar{\rho}\bar{J} = \rho J$, we find that

$$J = \bar{J}(1 + \varepsilon u_{j,j}) + O(\varepsilon^2), \quad \rho = \bar{\rho}(1 - \varepsilon u_{j,j}) + O(\varepsilon^2). \tag{3}$$

The internal constraints are taken in the form

$$\lambda^{(A)}(\mathbf{F}, \mathbf{X}) = 0, \quad A = 1, \dots, N, \tag{4}$$

and we stipulate that the number N of such constraints acting simultaneously shall satisfy $N \leqslant 5$ in order that the constraints shall not fully specify the strain.

The Cauchy stress $\boldsymbol{\sigma}$ measured in B_t and the strain energy $W(\mathbf{F}, \mathbf{X})$ per unit volume of B_0 of an unconstrained body are related by

$$\sigma_{ij} = J^{-1}\frac{\partial W}{\partial F_{iA}}F_{jA}.$$

It is well known that the existence of the constraints (4) in $\mathscr{B}$ gives rise to internal forces which contribute to $\boldsymbol{\sigma}$ a linear combination of reaction tensors $\mathbf{N}^{(A)}$, $A = 1, \dots, N$. Individually, these tensors do no work in any deformation compatible with the constraints, from which it follows that in B_t

$$N_{ij}^{(A)} = J^{-1}\frac{\partial \lambda^{(A)}}{\partial F_{iA}}F_{jA}, \quad A = 1, \dots, N.$$

The total Cauchy stress $\boldsymbol{\sigma}^*$ in the configuration B_t of the constrained body is then given by

$$\boldsymbol{\sigma}^* = \boldsymbol{\sigma}^c + \sum_A \alpha^{(A)} \mathbf{N}^{(A)} = J^{-1} \frac{\partial W^*}{\partial \mathbf{F}} \mathbf{F}^T \tag{5}$$

in which

$$W^* = W + \sum_A \alpha^{(A)} \lambda^{(A)}$$

is an augmented strain energy, $\boldsymbol{\sigma}^c$ denotes the constitutive part of the stress determined from the strain energy $W(\mathbf{F}, \mathbf{X})$ by

$$\sigma^c_{ij} = J^{-1} \frac{\partial W}{\partial F_{iA}} F_{jA}$$

and the

$$\alpha^{(A)} = \bar{\alpha}^{(A)}(\bar{\mathbf{x}}) + \varepsilon \beta^{(A)}(\mathbf{x}, t), \quad A = 1, \ldots, N, \tag{6}$$

are arbitrary scalar functions of position and time not directly dependent on the deformation gradient $\mathbf{F}$. The quantity $\bar{\alpha}^{(A)}(\bar{\mathbf{x}})$ in (6) is that part of the scalar multiplier that arises from $\bar{B}$, whilst $\varepsilon\beta^{(A)}(\mathbf{x}, t)$ is the contribution arising from the transition from $\bar{B}$ to B_t.

The total elastic modulus is defined by

$$d^*_{ijkl} = J^{-1} F_{jA} F_{lC} \frac{\partial^2 W^*}{\partial F_{iA} \partial F_{kC}} = d^c_{ijkl} + \sum_A \alpha^{(A)} d^{(A)}_{ijkl} \tag{7}$$

in which $\mathbf{d}^c$ and the $\mathbf{d}^{(A)}$ are defined by replacing W^* in $(7)_1$ in turn by W and the $\lambda^{(A)}$. The tensor $\mathbf{d}^*$ has the symmetries

$$d^*_{ijkl} = d^*_{klij}$$

because it is derived from the potential W^* but in general it possesses no others.

From (5) we may express the total Cauchy stress $\boldsymbol{\sigma}^*$ in B_t in terms of the total Cauchy stress $\bar{\boldsymbol{\sigma}}^*$ in $\bar{B}$ by expanding $\boldsymbol{\sigma}^c$ and $\mathbf{N}^{(A)}$ to $O(\varepsilon)$ using (2), $(3)_1$, (6) and the definition (7):

$$\sigma^*_{ij} = \bar{\sigma}^*_{ij}(1 - \varepsilon u_{l,l}) + \varepsilon\{d^*_{ijkl} u_{k,l} + \bar{\sigma}^*_{il} u_{j,l}\} + \varepsilon \sum_A \bar{N}^{(A)}_{ij} \beta^{(A)}. \tag{8}$$

The symmetry of (8) is not immediately apparent but may be demonstrated as follows. The principle of objectivity requires the strain energy W and the constraint function $\lambda^{(A)}$ to be functions of the right Cauchy–Green strain tensor $\mathbf{C} = \mathbf{F}^T\mathbf{F}$ rather than merely $\mathbf{F}$. We then find that

$$\sigma^*_{ij} = 2J^{-1} F_{iA} F_{jB} \frac{\partial W^*}{\partial C_{AB}}$$

replaces $(5)_2$ and that (7) is replaced by

$$d^*_{ijkl} = \delta_{ik} \sigma^*_{jl} + c^*_{ijkl} \tag{9}$$

in which

$$c^*_{ijkl} = 4J^{-1} F_{iA} F_{jB} F_{kC} F_{lD} \frac{\partial^2 W^*}{\partial C_{AB} \partial C_{CD}}.$$

It is clear from this equation that $\mathbf{c}^*$ possesses all the symmetries

$$c^*_{ijkl} = c^*_{klij} = c^*_{jikl}$$

associated with the usual fourth-order linear elasticity tensor. Indeed, it is clear from (9) that if there is no prestress σ^*_{ji} then $\mathbf{d}^*$ and $\mathbf{c}^*$ coincide, and that both must coincide with the linear elasticity tensor. On substituting (9) into (8) we obtain

$$\sigma^*_{ij} = \bar{\sigma}^*_{ij}(1-\varepsilon u_{l,l}) + \varepsilon\{\bar{\sigma}^*_{il}u_{j,l} + \bar{\sigma}^*_{jl}u_{i,l}\} + \varepsilon c^*_{ijkl}u_{k,l} + \varepsilon \sum_A N^{(A)}_{ij}\beta^{(A)},$$

and the symmetry of the Cauchy stress is now apparent.

The equations of equilibrium in $\bar{B}$ and of motion in B_t are

$$\bar{\sigma}^*_{ij,j} = 0, \quad \rho\ddot{x}_i = \frac{\partial \sigma^*_{ij}}{\partial x_j}, \tag{10}$$

respectively. The latter may be rewritten

$$\rho\varepsilon\ddot{u}_i = (\delta_{jk} - \varepsilon u_{k,j})\sigma^*_{ij,k}$$

into which we now substitute for ρ and σ^*_{ij} from $(3)_2$ and (8), respectively. The O(1) term in ε vanishes because of $(10)_1$. We ignore $O(\varepsilon^2)$ terms and cancel the factor ε from the $O(\varepsilon)$ terms to obtain, after a certain amount of manipulation, the equations of motion for small-amplitude vibrations superimposed upon a state of finite, static strain:

$$\bar{\rho}\ddot{u}_i = \{\bar{d}^*_{ijkl}u_{k,l}\}_{,j} + \left\{\sum_A \bar{N}^{(A)}_{ij}\beta^{(A)}\right\}_{,j}. \tag{11}$$

The quantities $\bar{\rho}$, $\bar{\mathbf{d}}^*$, $\bar{\alpha}^{(A)}$ and $\bar{\mathbf{N}}^{(A)}$ are all evaluated in the configuration $\bar{B}$ and so may depend upon $\bar{\mathbf{x}}$ but not t. Of course, the displacements $\mathbf{u}$ and the incremental scalar multipliers $\beta^{(A)}$ depend upon both $\bar{\mathbf{x}}$ and t.

The constraints (4) apply in both $\bar{B}$ and B_t and so by expanding them as far as the $O(\varepsilon)$ terms we obtain the further equations

$$\bar{N}^{(A)}_{ij}u_{i,j} = 0, \quad A = 1, \ldots, N. \tag{12}$$

Thus (11) and (12) constitute a set of $3+N$ partial differential equations for the $3+N$ unknown functions $\mathbf{u}$ and $\beta^{(A)}$. (We do not consider initial and boundary conditions because for the most part we shall be confining attention to sinusoidal plane waves propagating through an unbounded material.)

Equations (11) are valid for arbitrary forms of small vibration superimposed upon an arbitrary state of prestrain and prestress of an elastic material of arbitrary anisotropy suffering an unspecified number of constraints of fully general form. In the particular case of a material suffering the constraints of both incompressibility and inextensibility eqns (11) reduce to those given by Scott and Hayes [1976, eqns (3.26)]. For an unconstrained material eqns (11) reduce to those given by Toupin and Bernstein [1961, eqns (2.19)]. If the prestrain and prestress are homogeneous so that $\bar{\rho}$, $\bar{\mathbf{d}}^*$, $\bar{\alpha}^{(A)}$ and $\bar{\mathbf{N}}^{(A)}$ are constant then (11) reduce eqns to

$$\bar{\rho}\ddot{u}_i = \bar{d}^*_{ijkl}u_{k,jl} + \sum_A \bar{N}^{(A)}_{ij}\beta^{(A)}_{,j} \tag{13}$$

which are the equations of Chadwick *et al.* [1985, eqns (3.14)].

From now on we specialize to the case of sinusoidal vibrations

$$u_i = p_i e^{i\omega(v^{-1}\mathbf{n}\cdot\mathbf{x}-t)}, \quad \beta^{(A)} = i\omega b^{(A)} e^{i\omega(v^{-1}\mathbf{n}\cdot\mathbf{x}-t)}, \tag{14}$$

representing a plane wave of frequency ω travelling with speed v through an infinite medium in the direction of the unit vector $\mathbf{n}$ and having constant polarization $\mathbf{p}$. The constants $b^{(A)}$ are the amplitudes of the constraint stresses. On substituting (14) into (13) and (12) and cancelling common factors we get the propagation conditions

$$\mathbf{Q}^*(\mathbf{n})\mathbf{p}+v\sum_A \mathbf{v}^{(A)}(\mathbf{n})b^{(A)} = \bar{\rho}v^2\mathbf{p}, \tag{15}$$

$$\mathbf{v}^{(A)}(\mathbf{n})\cdot\mathbf{p} = 0, \quad A = 1,\ldots,N, \tag{16}$$

in which the symmetric acoustic tensor $\mathbf{Q}^*(\mathbf{n})$ and the constraint vectors $\mathbf{v}^{(A)}(\mathbf{n})$ are defined by

$$Q^*_{ij}(\mathbf{n}) = \bar{d}^*_{ijkl}n_j n_l, \quad v_i^{(A)}(\mathbf{n}) = \bar{N}_{ij}^{(A)}n_j, \quad A = 1,\ldots,N. \tag{17}$$

We define $\mathbf{Q}^c(\mathbf{n})$ and $\mathbf{Q}^{(A)}(\mathbf{n})$ by replacing $\mathbf{d}^*$ in $(17)_1$ by $\mathbf{d}^c$ and $\mathbf{d}^{(A)}$, respectively.

We now specialize to a material in which two constraints are acting. For those $\mathbf{n}$ such that the two constraint vectors $\mathbf{v}^{(1)}$ and $\mathbf{v}^{(2)}$ are linearly independent we see from (16) that the unit polarization vector is given by

$$\mathbf{p} = \hat{\boldsymbol{\mu}}, \quad \hat{\boldsymbol{\mu}} \equiv \frac{\mathbf{v}^{(2)}\wedge\mathbf{v}^{(1)}}{|\mathbf{v}^{(2)}\wedge\mathbf{v}^{(1)}|},$$

and that (15) then yields an explicit expression for the squared wave speed:

$$\bar{\rho}v^2 = \hat{\boldsymbol{\mu}}\cdot\mathbf{Q}^*(\mathbf{n})\hat{\boldsymbol{\mu}}.$$

If the constraint vectors fail to be linearly independent but span a space of dimension one then we denote by $\hat{\mathbf{v}}(\mathbf{n})$ a unit vector that spans this pace. On writing

$$\mathbf{P} = \mathbf{1}-\hat{\mathbf{v}}(\mathbf{n})\otimes\hat{\mathbf{v}}(\mathbf{n})$$

we find that (15) and (16) reduce to the eigenvalue problem

$$\{\bar{\rho}v^2\mathbf{1}-\mathbf{P}\mathbf{Q}^*\}\mathbf{p} = \mathbf{0}, \quad \hat{\mathbf{v}}\cdot\mathbf{p} = 0, \tag{18}$$

having the same form as the propagation condition of a singly constrained material. The eigenvalue $\bar{\rho}v^2 = 0$ of $(18)_1$ is spurious since the corresponding eigenvector $\mathbf{p}$ does not satisfy $(18)_2$. The other two eigenvalues possess mutually orthogonal eigenvectors that also satisfy $(18)_2$. These results were first derived for general constraints by the present author [see Scott (1975), Section 8] in the context of acceleration wave theory.

For those $\mathbf{n}$ for which both constraint vectors vanish we see that (16) becomes redundant and (15) reduces to

$$\{\bar{\rho}v^2\mathbf{1}-\mathbf{Q}^*(\mathbf{n})\}\mathbf{p} = \mathbf{0},$$

the same form as the propagation condition of an unconstrained material.

In order to model an idealized fibre-reinforced material we shall consider a material that suffers two constraints simultaneously, namely incompressibility and inextensibility along the fibre direction.

Incompressibility

We take this constraint in the form $\lambda^{(1)}(\mathbf{F}) \equiv \det \mathbf{F} - 1 = 0$ so that we have

$$\mathbf{N}^{(1)} = \mathbf{1}, \quad \boldsymbol{\nu}^{(1)}(\mathbf{n}) = \mathbf{n}, \quad d^{(1)}_{ijkl} = \delta_{ij}\delta_{kl} - \delta_{il}\delta_{jk}, \quad \mathbf{Q}^{(1)}(\mathbf{n}) = \mathbf{0}. \tag{19}$$

The corresponding multiplier $\alpha^{(1)}$ may be interpreted as the negative of an arbitrary pressure needed to maintain the constraint of incompressibility.

Inextensibility

Let the fibres in the undeformed solid be aligned with the constant unit vector $\mathbf{A}$ so that in the deformed configuration they are aligned with the fixed direction $\mathbf{a} = \mathbf{FA}$. The constraint of inextensibility may then be written as $\lambda^{(2)}(\mathbf{F}) \equiv \frac{1}{2}(\mathbf{a}\cdot\mathbf{a} - 1) = 0$, so that

$$\mathbf{N}^{(2)} = J^{-1}\mathbf{a}\otimes\mathbf{a}, \quad \boldsymbol{\nu}^{(2)}(\mathbf{n}) = J^{-1}(\mathbf{a}\cdot\mathbf{n})\mathbf{a}, \quad d^{(2)}_{ijkl} = J^{-1}\delta_{ik}a_j a_l,$$
$$\mathbf{Q}^{(2)}(\mathbf{n}) = J^{-1}(\mathbf{a}\cdot\mathbf{n})^2\mathbf{1}. \tag{20}$$

The multiplier $\alpha^{(2)}$ may be interpreted as an arbitrary tension along the fibre needed to maintain the constraint of inextensibility.

3. THE SLOWNESS SURFACE OF AN IDEALIZED FIBRE-REINFORCED MATERIAL

On specializing to an elastic material that is both incompressible and inexensible the propagation conditions (15) and (16) become

$$\mathbf{Q}^c(\mathbf{n})\mathbf{p} - v p_1 \mathbf{n} + v T_1(\mathbf{a}\cdot\mathbf{n})\mathbf{a} = (\bar{\rho}v^2 - T(\mathbf{a}\cdot\mathbf{n})^2)\mathbf{p}, \tag{21}$$

$$\mathbf{n}\cdot\mathbf{p} = 0, \quad (\mathbf{a}\cdot\mathbf{n})(\mathbf{a}\cdot\mathbf{p}) = 0, \tag{22}$$

in which the symbol for the multiplier $\bar{\alpha}^{(2)}$ has been replaced by T in recognition of its role as a tension along the fibre and the quantity $\bar{\alpha}^{(1)}$ does not appear. The constants $b^{(1)}$ and $b^{(2)}$ have been replaced by $-p_1$ and T_1, respectively, for similar reasons. The factor J^{-1} has been omitted from certain terms since the constraint of incompressibility implies that its value is unity.

For wave propagation in any direction normal to the fibre, i.e. satisfying $\mathbf{a}\cdot\mathbf{n} = 0$, we see from $(20)_2$ that $\boldsymbol{\nu}^{(2)}(\mathbf{n}) = \mathbf{0}$ so that the two constraint vectors fail to be linearly independent. Every direction in this plane whose normal is a constitutes an exceptional direction for wave propagation and $(22)_2$ makes no restriction on the wave polarization $\mathbf{p}$. Then (18), (21) and (22) reduce to

$$\{\bar{\rho}v^2\mathbf{1} - \mathbf{P}\mathbf{Q}^c(\mathbf{n})\}\mathbf{p} = \mathbf{0}, \quad \mathbf{n}\cdot\mathbf{p} = 0, \tag{23}$$

for all $\mathbf{n}$ such that $\mathbf{a}\cdot\mathbf{n} = 0$ where now $\mathbf{P} = \mathbf{1} - \mathbf{n}\otimes\mathbf{n}$.

For wave propagation in the fibre direction $\mathbf{a}$ we see from $(19)_2$ and $(20)_2$ that $\boldsymbol{\nu}^{(1)}(\mathbf{n})$ and $\boldsymbol{\nu}^{(2)}(\mathbf{n})$ are parallel since $\mathbf{n} = \mathbf{a}$. Then $(22)_1$ and $(22)_2$ both supply the same restriction upon $\mathbf{p}$ and in this exceptional direction of wave propagation (18), (21) and (22) reduce to

$$\{(\bar{\rho}v^2 - T)\mathbf{1} - \mathbf{P}\mathbf{Q}^c(\mathbf{a})\}\mathbf{p} = \mathbf{0}, \quad \mathbf{a}\cdot\mathbf{p} = 0, \tag{24}$$

where $\mathbf{P}$ is as in (23) but now $\mathbf{n} = \mathbf{a}$.

In all other cases, where the wave direction $\mathbf{n}$ is neither parallel to nor normal to the fibre $\mathbf{a}$, eqns (22) provide two independent restrictions on the wave polarization $\mathbf{p}$ and so determine it uniquely, up to sign, as

$$\mathbf{p} = \hat{\boldsymbol{\mu}} \equiv \frac{\mathbf{a} \wedge \mathbf{n}}{|\mathbf{a} \wedge \mathbf{n}|}. \tag{25}$$

From (21) we obtain the entirely explicit expression

$$\bar{\rho}v^2 = T(\mathbf{a}\cdot\mathbf{n})^2 + \hat{\boldsymbol{\mu}}\cdot\mathbf{Q}^c(\mathbf{n})\hat{\boldsymbol{\mu}}, \tag{26}$$

for the squared wave speeds in the non-exceptional directions $\mathbf{n}$. The slowness surface is a centro-symmetric surface of one sheet formed by the locus of the slowness

$$\mathbf{s} = v^{-1}\mathbf{n}$$

for all non-exceptional $\mathbf{n}$ with wave speed v given in terms of $\mathbf{n}$ by (26). On dividing (26) by v^2 we see that the equation of the slowness surface may be written in the form

$$\bar{\rho} = \{Ta_i a_l + d^c_{ijkl}\hat{\mu}_j\hat{\mu}_k\} s_i s_l. \tag{27}$$

Consider any plane cross-section of the slowness surface (27) that contains the fibre direction $\mathbf{a}$ and the origin. For all non-exceptional $\mathbf{n}$ contained in this cross-section, $\hat{\boldsymbol{\mu}}$ is a unit normal to the cross-section and is therefore independent of the choice of $\mathbf{n}$ within that cross-section. Thus (27) represents a quadric surface for each fixed $\hat{\boldsymbol{\mu}}$ and so its intersection with the plane $\hat{\boldsymbol{\mu}}\cdot\mathbf{s} = 0$ is a plane conic section. We shall assume that the material satisfies the condition of strong ellipticity, so that $\hat{\boldsymbol{\mu}}\cdot\mathbf{Q}^c(\mathbf{n})\hat{\boldsymbol{\mu}} > 0$. However, this is not in general sufficient to force a constrained material to have positive squared wave speeds; for example, if $T < 0$ and $|T|$ is large enough then the right-hand side of (26) is negative. Chen and Gurtin (1974) have associated such values of T in an inextensible solid with buckling modes. In order to restrict attention to propagating modes we shall admit only those arbitrary tensions T that give rise to a positive value for the squared wave speed. Since $\bar{\rho}v^2$ is clearly bounded when viewed as a function of $\mathbf{n}$ (and has been restricted to be positive), it follows that the plane conic section in the plane $\hat{\boldsymbol{\mu}}\cdot\mathbf{s} = 0$ is either elliptical or circular. This conclusion is borne out by parts (v) and (vi) of Figs 2 and 3.

Scott and Hayes (1976, Section 7) have shown that the slowness surface of an unstrained idealized fibre-reinforced material is necessarily ellipsoidal. Despite our conclusions in the previous paragraph, this result does not carry over to the prestrained material. In general, it is only those cross-sections of the slowness surface containing the fibre direction $\mathbf{a}$ that are elliptical [see parts (i)–(iii) of Figs 2 and 3 for cross-sections that are not elliptical]. Also, the directions $\mathbf{n} = \pm\mathbf{a}$ are exceptional for the prestrained material but not for the unstrained material.

We now discuss the exceptional directions of wave propagation.

Exceptional case (i) $\mathbf{n} = \mathbf{a}$

Let us consider the limit $\mathbf{n} \to \mathbf{a}$ as $\mathbf{n}$ varies in the cross-section $\hat{\boldsymbol{\mu}}\cdot\mathbf{s} = 0$ of the slowness surface where $\hat{\boldsymbol{\mu}}$ is any given unit vector satisfying $\hat{\boldsymbol{\mu}}\cdot\mathbf{a} = 0$. Then (26) and (27) remain well defined in such a limit and so (26) will return a value for the squared wave speed even when $\mathbf{n} = \mathbf{a}$. However, taking a different cross-section (i.e. selecting a different unit vector $\hat{\boldsymbol{\mu}}$ such that $\hat{\boldsymbol{\mu}}\cdot\mathbf{a} = 0$) will in general result in a different limiting value for the squared wave speed in the same direction $\mathbf{n} = \mathbf{a}$. This is borne out by parts (v) and (vi) of Figs 2 and 3 in which it is seen that the two elliptical cross-sections shown cut the fibre direction $\mathbf{a}$ at different points.

For wave propagation in the fibre direction, eqn (24) gives in general two distinct wave speeds with polarizations that are orthogonal to each other and to the fibre direction $\mathbf{a}$. It is natural to enquire about the connection between these two waves and the wave given by (26) as $\mathbf{n} \to \mathbf{a}$ in any fixed plane (with normal $\hat{\boldsymbol{\mu}}$) that contains the fibre direction $\mathbf{a}$. We shall see in the next paragraph that the two squared wave speeds provided by (24) are in fact the extreme values of (26), evaluated at $\mathbf{n} = \mathbf{a}$, as the unit vector $\hat{\boldsymbol{\mu}}$ varies over all directions satisfying $\mathbf{a} \cdot \hat{\boldsymbol{\mu}} = 0$.

To prove this last assertion we contract $(24)_1$ with $\mathbf{p}$ and attempt to extremize the following expressions for the squared wave speed:

$$\bar{\rho} v^2 = F(\mathbf{p}) \equiv T\mathbf{p} \cdot \mathbf{p} + \mathbf{p} \cdot \mathbf{Q}^{\xi}(\mathbf{a})\mathbf{p} + \lambda(\mathbf{p} \cdot \mathbf{p} - 1) + 2\zeta \mathbf{a} \cdot \mathbf{p},$$

where λ and ζ are Lagrange multipliers corresponding to the constraints $\mathbf{p} \cdot \mathbf{p} = 1$ and $\mathbf{a} \cdot \mathbf{p} = 0$, respectively. In the usual way we equate $\partial F/\partial \mathbf{p}$ to zero:

$$T\mathbf{p} + \mathbf{Q}^{\xi}(\mathbf{a})\mathbf{p} + \lambda \mathbf{p} + \zeta \mathbf{a} = \mathbf{0}, \tag{28}$$

and then contract in turn with $\mathbf{p}$ and $\mathbf{a}$ to obtain $\lambda = -T - \mathbf{p} \cdot \mathbf{Q}^{\xi}(\mathbf{a})\mathbf{p} = -\bar{\rho} v^2$ and $\zeta = -\mathbf{a} \cdot \mathbf{Q}^{\xi}(\mathbf{a})\mathbf{p}$. On inserting these values of the Lagrange multipliers back into (28) we find that (24) results. Thus the two squared wave speeds in the exceptional case $\mathbf{n} = \mathbf{a}$, given by (24), coincide with the extrema of (26).

This property is illustrated by parts (v) and (vi) of each of Figs 2 and 3 in which the elliptical cross-section cuts the direction $\mathbf{n} = \mathbf{a}$ at or between the two horizontal bars which themselves indicate the two slownesses corresponding to the eigenvalues of (24).

We are now in a position to determine the nature of the slowness surface in the vicinity of the singular exceptional direction $\mathbf{n} = \mathbf{a}$. To this end we rewrite (26) as

$$\bar{\rho} v^2 = \{T(\mathbf{a} \cdot \mathbf{n})^2 \delta_{ik} + d^{\xi}_{ijkl} n_j n_l\} \hat{\mu}_i \hat{\mu}_k \tag{29}$$

and permit the wave direction $\mathbf{n}$ to describe a cone about the fibre direction with semi-apex angle δ satisfying $0 < \delta \ll 1$. The quantity within braces in (29) is then equal to $T\delta_{ik} + d^{\xi}_{ijkl} a_j a_l + O(\delta)$ and so varies very little as this cone is described whereas the unit vector $\hat{\boldsymbol{\mu}}$ describes the unit circle $\mathbf{a} \cdot \hat{\boldsymbol{\mu}} = 0$. Consequently, $v^{-1}\hat{\boldsymbol{\mu}}$ describes an ellipse and the slowness $\mathbf{s} = v^{-1}\mathbf{n}$ varies rapidly as this cone of small apex angle is described, even though $\mathbf{n}$ itself varies very little. The slowness $\mathbf{s} = v^{-1}\mathbf{n}$ follows curve C on Fig. 1, approaching the slowness corresponding to each eigenvalue of (24)—the maximum and minimum slownesses—exactly twice in one circuit.

Exceptional case (ii) $\mathbf{a} \cdot \mathbf{n} = 0$

In addition to the exceptional direction of propagation $\mathbf{n} = \mathbf{a}$, each cross-section $\hat{\boldsymbol{\mu}} \cdot \mathbf{s} = 0$ (where $\hat{\boldsymbol{\mu}}$ is, as before, a given unit vector such that $\mathbf{a} \cdot \hat{\boldsymbol{\mu}} = 0$) of the slowness surface (27) contains two further exceptional directions of propagation and these satisfy $\mathbf{a} \cdot \mathbf{n} = 0$; they are $\mathbf{n} = \pm \hat{\boldsymbol{\mu}} \wedge \mathbf{a}$. They share the same squared wave speed given by (26), so that as $\mathbf{n} \to \pm \hat{\boldsymbol{\mu}} \wedge \mathbf{a}$ through directions lying in the cross-section $\hat{\boldsymbol{\mu}} \cdot \mathbf{s} = 0$ the limiting squared wave speed given by (26) is approached in a continuous manner. However, this limiting wave speed does not in general coincide with either of the wave speeds predicted by (23) for the exceptional directions $\mathbf{n}$ of propagation satisfying $\mathbf{a} \cdot \mathbf{n} = 0$. However, we may use the

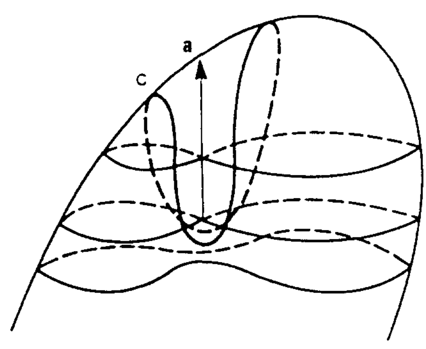

Fig. 1. The slowness surface of an incompressible, inextensible material near $\mathbf{n} = \mathbf{a}$.

methods of the previous sub-section to show that for each $\mathbf{n}$ such that $\mathbf{a}\cdot\mathbf{n} = 0$ the limiting squared wave speed given by (26) lies between the two eigenvalues of (23).

These phenomena are illustrated in part (iv) of each of Figs 2 and 3 where the broken curve representing the slowness derived from the limit of (26) lies between the two full curves representing the two slownesses derived from the eigenvalues of (23).

4. NUMERICAL ILLUSTRATIONS

None of the theoretical results of this paper depend on the precise form of the strain energy W which could therefore represent an incompressible, inextensible material of arbitrary elastic anisotropy. However, it is customary to model an idealized fibre-reinforced material as an incompressible isotropic matrix reinforced by inextensible fibres so that the continuum model of the resulting solid is that of an incompressible material that is transversely isotropic about the inextensible fibre direction. In this case the strain energy depends only on the three invariants

$$j_1 = \operatorname{tr}\mathbf{C}, \quad j_2 = \tfrac{1}{2}\operatorname{tr}\mathbf{C}^2, \quad j_3 = \mathbf{A}\cdot\mathbf{C}^2\mathbf{A},$$

as explained by Spencer (1972). If we make the further assumption, merely to simplify the numerical calculations, that the fibres are sparsely dispersed throughout the matrix material then we may disregard the dependence of W on the invariant j_3.

For illustrative purposes, then, we adopt the strain energy

$$W = c_1(j_1-3)+c_2(\tfrac{1}{2}j_1^2-j_2-3)+c_3(j_2-\tfrac{3}{2}) \tag{30}$$

in which c_1, c_2 and c_3 are constants. If $c_3 = 0$, (30) represents a Mooney–Rivlin material and if also $c_2 = 0$, the material is neo-Hookean. As it stands, (30) is the most general form of the strain energy of an incompressible elastic solid if terms smaller than $(\lambda_i-1)^4$ and $(\lambda_i-1)^2(\lambda_j-1)^2$, $i, j = 1, 2, 3$, are ignored; the quantities λ_i are the principal stretches of the prestrain.

It is convenient to take Cartesian axes coincident with the principal axes of the left Cauchy–Green strain tensor $\mathbf{B} = \mathbf{F}\mathbf{F}^{\mathrm{T}}$, so that

$$\mathbf{B} = \lambda_1^2\mathbf{i}\otimes\mathbf{i}+\lambda_2^2\mathbf{j}\otimes\mathbf{j}+\lambda_3^2\mathbf{k}\otimes\mathbf{k},$$

where $\{\mathbf{i},\mathbf{j},\mathbf{k}\}$ constitutes a right-handed orthonormal triad codirectional with the principal axes of $\mathbf{B}$. It is also convenient to order the λ_i so that either λ_1, $\lambda_2 \geqslant 1$ and $\lambda_3 < 1$, or λ_1, $\lambda_2 \leqslant 1$ and $\lambda_3 > 1$ holds true. Unless $\lambda_1 = \lambda_2 = \lambda_3 = 1$, one of these two orderings is possible because of the constraint of incompressibility:

$$\lambda_1\lambda_2\lambda_3 = 1. \tag{31}$$

The constraint of inextensibility may be expressed as $\mathbf{a}\cdot\mathbf{B}^{-1}\mathbf{a} = 1$, which on writing $\mathbf{a} = a_1\mathbf{i}+a_2\mathbf{j}+a_3\mathbf{k}$ becomes

$$\frac{a_1^2}{\lambda_1^2}+\frac{a_2^2}{\lambda_2^2}+\frac{a_3^2}{\lambda_3^2} = 1. \tag{32}$$

We now write $\mathbf{a}$ in terms of spherical polar coordinates with $\mathbf{k}$ as the polar axis so that

$$a_1 = \sin\theta\cos\phi, \quad a_2 = \sin\theta\sin\phi, \quad a_3 = \cos\theta.$$

Then (32) yields the following expression for θ in terms of ϕ and the principal stretches:

$$\sin^2\theta = \frac{1-\dfrac{1}{\lambda_3^2}}{\dfrac{\cos^2\phi}{\lambda_1^2}+\dfrac{\sin^2\phi}{\lambda_2^2}-\dfrac{1}{\lambda_3^2}}. \tag{33}$$

If the λ_i are ordered as in the previous paragraph then for any value of ϕ it can be shown that (33) defines a real angle θ satisfying $0 \leqslant \theta \leqslant \pi/2$.

The predeformation and the inextensible fibre direction **a** are uniquely specified if λ_1, λ_2 (satisfying $\lambda_1, \lambda_2 \leqslant 1$ or $\lambda_1, \lambda_2 \geqslant 1$) and ϕ are specified because λ_3 may then be obtained from (31) and θ from (33).

It is convenient to introduce the right-handed orthonormal triad $\{\mathbf{a}, \mathbf{a}', \mathbf{a}''\}$ based on the fibre direction **a** and defined in terms of $\{\mathbf{i}, \mathbf{j}, \mathbf{k}\}$ by

$$\left.\begin{aligned}\mathbf{a}' &= \mathbf{i}\sin\phi - \mathbf{j}\cos\phi,\\ \mathbf{a}'' &= \mathbf{a}\wedge\mathbf{a}' = \mathbf{i}\cos\theta\cos\phi + \mathbf{j}\cos\theta\sin\phi - \mathbf{k}\sin\theta.\end{aligned}\right\} \tag{34}$$

We now discuss the cross-sections of the slowness surface presented in Figs 2 and 3 for given **B** and **a** and W of the form (30).

In each figure we see six cross-sections of the same slowness surface, each containing the origin and each drawn on the same scale. Each of the left-hand graphs, numbered (i)–(iii), is a cross-section containing a principal plane of **B**. The diagonal line, marked $\mathbf{a}^P$, is

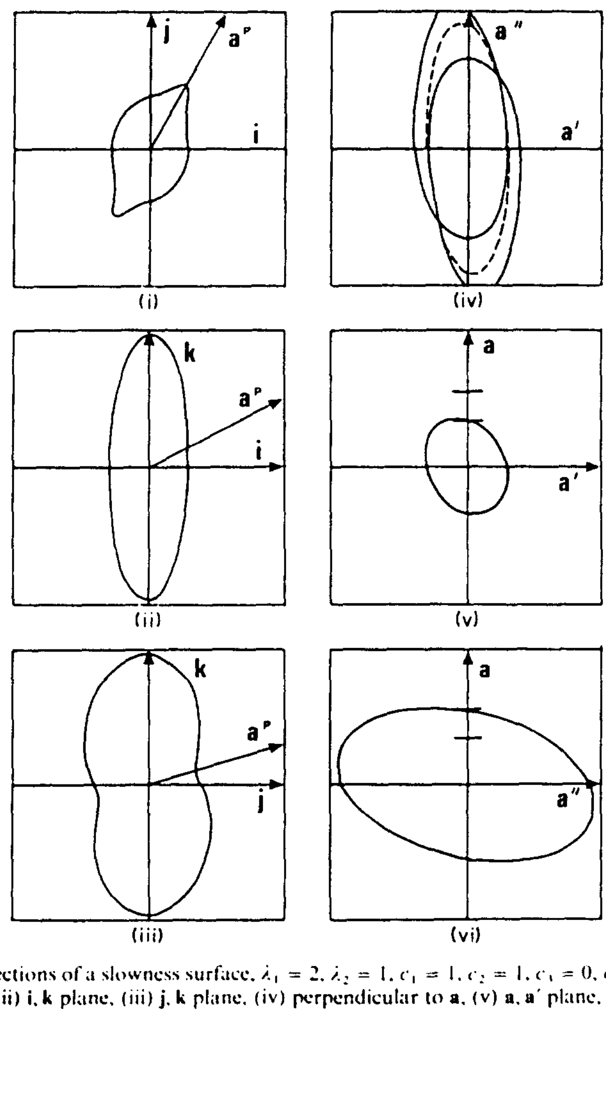

Fig. 2. Cross-sections of a slowness surface, $\lambda_1 = 2$, $\lambda_2 = 1$, $c_1 = 1$, $c_2 = 1$, $c_3 = 0$, $\phi = 60°$, $\theta \approx 76°$. (i) **i**, **j** plane, (ii) **i**, **k** plane, (iii) **j**, **k** plane, (iv) perpendicular to **a**, (v) **a**, **a**′ plane, (vi) **a**, **a**″ plane.

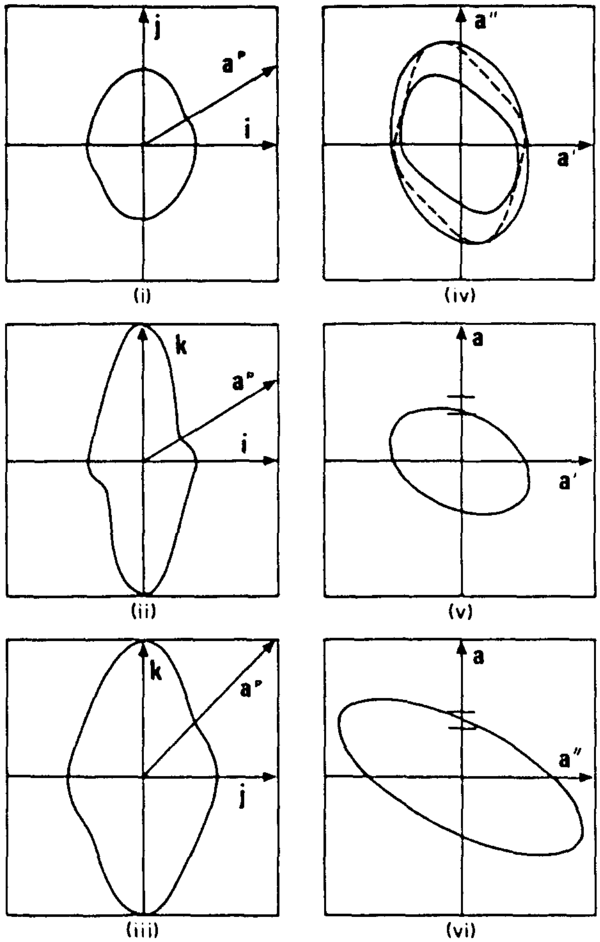

Fig. 3. Cross sections of a slowness surface, $\lambda_1 = 1.1$, $\lambda_2 = 1$, $c_1 = 1$, $c_2 = 1$, $c_3 = 0$, $\phi = 30^\circ$, $\theta \approx 63^\circ$, (i) $\mathbf{i}, \mathbf{j}$ plane, (ii) $\mathbf{i}, \mathbf{k}$ plane, (iii) $\mathbf{j}, \mathbf{k}$ plane, (iv) perpendicular to $\mathbf{a}$, (v) $\mathbf{a}, \mathbf{a}'$ plane, (vi) $\mathbf{a}, \mathbf{a}''$ plane.

the projection of the fibre direction onto the cross-section depicted. This projection is, of course, a different vector in each graph but for simplicity each has been denoted by the same symbol $\mathbf{a}^P$.

Each of the right-hand graphs, numbered (iv)–(vi), is a cross-section based on $\{\mathbf{a}, \mathbf{a}', \mathbf{a}''\}$. Part (iv) depicts the situation in the plane whose normal is $\mathbf{a}$, which is exceptional case (ii) discussed previously. The two full curves are those supplied by the analysis for the exceptional case which leads to the eigenvalue problem (23), whilst the broken curve represents the value of the slowness obtained by taking the limit as $\mathbf{n}$ approaches a direction such that $\mathbf{a} \cdot \mathbf{n} = 0$ of the single-sheeted slowness surface (26) valid for all non-exceptional directions.

Parts (v) and (vi) depict cross-sections containing the fibre direction $\mathbf{a}$ and so the analysis of exceptional case (i) applies. The fibre direction is aligned with the vertical axis and the horizontal bars mark the slownesses obtained from the analysis of the exceptional case, see (24). The curve depicted is the elliptical cross-section predicted by the analysis for non-exceptional directions of wave propagation. In accord with the analysis for exceptional case (i), it can be seen that the ellipse cuts the fibre direction at a different point in each of parts (v) and (vi) but that each point is at or between the horizontal bars obtained from (24).

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